A Differential Tree Approach to Price Path-Dependent American Options using Malliavin Calculus

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Abstract. We propose a recursive schemes to calculate backward the values of conditional expectations of functions of path values of Brownian motion. This scheme is based on the Clark-Ocone formula in discrete time. We suggest an algorithm based on our scheme to effectively calculate the price of American options on securities with path-dependent payoffs. For problems where the path-dependence comes only from the path-dependence of the state variables our method is less subject to the curse of dimensionality observed in all other methods.

Keywords: Malliavin calculus, Monte Carlo methods, optimal stopping.
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INTRODUCTION

We consider a finite horizon optimal stopping problem. The uncertainty is described by the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathscr{P})$, where $\{\mathscr{F}_t\}$ is the filtration generated by one-dimensional[1] Brownian motion W, and $\mathscr{P}$ is the risk-neutral measure. An American (more accurately a Bermudan) option can be exercised at time steps $\Delta, \ldots, M\Delta$. When exercised at time $m\Delta$, the option holder receives the $\mathscr{F}_{m\Delta}$-adapted payoff $h(m\Delta)$ which is a function of values taken by Brownian motion along its path:

$$h(m\Delta) = F(W(\Delta), \ldots, W(m\Delta), m)$$

Without loss of generality we assume the interest rate is zero. By standard arguments, the price at time $m\Delta$ of the option, which we call $V(m\Delta)$ is obtained by backward induction:

$$V(M\Delta) = h(M\Delta) \tag{1}$$

$$V(m\Delta) = \max\{h(m\Delta), E[V((m+1)\Delta) | \mathscr{F}_{m\Delta}]\} \quad 0 \leq m \leq M-1 \tag{2}$$

This simple overall strategy has been plagued numerically by what many authors call the "curse of dimensionality", namely that in most models, the dimension of the state variables generating the information is too large to calculate an approximation $\hat{C}$ of the continuation value:

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[1] All the results carry to higher dimension, with higher notational cost.

$$\hat{C}(m\Delta) \simeq E[V((m+1)\Delta)|\mathscr{F}_{m\Delta}] \quad (3)$$

Optimal stopping problems are usually analyzed as Markov decision problems. In that formulation, the value of the state variables at time t is sufficient to determine the continuation value of the option at time t. By augmenting the state space with lagged values of the state variables if necessary, the information contained in the (augmented) state variables at time t contains all the previous information, thus resulting often in a very large state space. This curse of dimensionality affects not only Markov chain implementations but also direct applications ("Monte Carlo on Monte Carlo") of the Monte Carlo method to calculate conditional expectations. Thus several techniques have been devised to improve Monte Carlo. Currently, practitioners seem to favor "regression-based" methods, such as the Tsitsiklis and Van Roy algorithm (TVR) [1] or the Longstaff and Schwartz algorithm (LS) [2]. Apparently, regression algorithms solve the "curse of dimensionality" problem: the number of scenarios does not increase exponentially with the number of time steps as in the "Monte Carlo on Monte Carlo" method. However, like in other methods, it renters the scene in disguise because the number of approximating functions required increases with the dimensionality of the state, as has been documented by Glasserman and Yu [3] and Egloff [4].

We developed in Schellhorn [5] a "backward Taylor expansion" using the Clark-Ocone formula, which allowed to calculate backward recursively conditional expectations of functionals based on the expected value of the Malliavin derivatives of all orders of the same functional at the next time step. We propose in this article a numerical scheme that allows the calculation of the latter without having an explicit analytical representation. We still need to add scenarios to numerically calculate derivatives. Indeed, our implementation uses a differential non-recombining tree, so we call it the DNRT method. In other terms, for scenario reaching a particular node, we need one extra scenario to calculate the first derivative, two scenarios for the second derivative of the conditional expectation and so forth.

For problems where the state variable is not path-dependent, the DNRT approach does not result in a substantial improvement over numerical approaches to solve PDEs, or, equivalently, a (non-recombining) tree probabilistic approach[2]. There are important problems in finance where the underlying, or state variable is itself path-dependent. There are thus potentially two levels of path-dependency:

- path-dependency of the underlying as a functional of Brownian motion
- path-dependency of the option payoff as a functional of the underlying variable

Two typical examples of these two levels of path-dependence in finance is the pricing of mortgage backed securities and swaptions, where the interest rate is modelled according to the Heath-Jarrow-Morton framework, such as the Libor (market) model. Since the PDE approach requires Markovian state variables, this problem needs to be transformed. There are two ways to transform this particular problem, i.e. to define a

[2] It is well-known that the trinomial tree approach is a probabilistic equivalent of the explicit finite difference method to solve parabolic PDEs.

vector of Markov state variables. The first method is stack the previous values of the interest rate into a state vector. The second one is to take the vector of forward rates as state variables. Since it is usually easier to calculate the payoff of an option as a function of the forward rates, the latter approach is usually adopted. However, this augmentation of the state space usually results in a huge numerical problem, which is impossible to handle by PDE methods[3]. Thus, the usual way to handle interest-rate derivatives is by Monte Carlo. A typical implementation for swaptions is given in Longstaff, Santa-Clara, and Schwartz [7].

However, some interest-rate derivatives, like caps and captions, depend only on the short interest rate. For these derivatives, our DNRT implementation will avoid the curse of dimensionality inherent in all previous applications of the Heath-Jarrow-Morton model.

To continue on our classification of path-dependence, most equity options use currently path-independent models of underlying variables, even if the cash flows are path-dependent. The Asian option which we consider in this paper is an example of such a security. Table 1 summarizes our classification of path-dependence.

	STATE VARIABLE	
Payoff	Path-dependent	Path-independent
Path-dependent	MBS	Asian option
Path-independent	cap, caption	Plain vanilla options

Table 1: Classification of path-dependence

We showed in Schellhorn and Morris [8] an earlier version of our algorithm, which was capable of valuing American securities where the path-dependence was limited to payoffs. That algorithm completely bypassed the curse of dimensionality. We show in this article how to price any American security, with path-dependence in both payoffs and state variables. We emphasize that our algorithm is subject to only one curse of dimensionality, the one that arises from the path-dependence of payoffs, and not the path-dependence in the state variable. We apply our algorithm to the pricing of Asian options, the prototypical example of path-dependent securities.

Fournie, Lasry, Lebuchoux and Lions [9, 10] show how to use the Malliavin integration by parts formula to numerically calculate this conditional expectation as the ratio of two well-behaved unconditional expectations. Bally, Caramellino and Zanette [11] extended these results, and incorporated them in an algorithm to price and hedge American options. Fujiwara and Kijima [12] extend their work to American options with a mild path-dependency. We found it difficult to generalize their result to heavily path-dependent options. Our approach uses different tools from theirs.

[3] A typical calculation of the price of MBS requires the discretization of the time axis into 360 time-steps, one for every month, see Schellhorn and Kidani [6].

RECURSIVE CALCULATION OF CONDITIONAL EXPECTATIONS

We denote by $\mathscr{F}^d \subset \mathscr{F}$ the flitration generated by the discrete values of Brownian motion $W(\Delta),..,W(M\Delta)$. The problem is to calculate the conditional expectation at time $i\Delta$ of a $\mathscr{F}^d_{(i+1)\Delta}$-measurable random variable F. We write $D^j_t F$ for the Malliavin derivative at time t of a functional F. The following proposition was conjectured in Schellhorn (2007). Its proof is contained in Schellhorn and Morris [13].

Proposition 1 *Let F be a functional for which the Malliavin derivative is well-defined. Then, for $i < m$:*

$$E[F|\mathscr{F}_{i\Delta}] = E[F|\mathscr{F}_{(m+1)\Delta}] - \sum_{k=i}^{m}\sum_{j=1}^{\infty}(-1)^{j+1} g_{k,j} E[D^j_{(k+1)\Delta}F|\mathscr{F}_{(k+1)\Delta}] \qquad (4)$$

where $g_{k,j}$ are functions which depend only on $W_{(k+1)\Delta} - W_{k\Delta}$ and deterministic terms, and do not depend on F . We have for instance:

$$\begin{aligned} g_{k,1} &= W_{(k+1)\Delta} - W_{k\Delta} \\ g_{k,2} &= \frac{1}{2}(W_{(k+1)\Delta} - W_{k\Delta})^2 + \frac{1}{2}\Delta \\ g_{k,3} &= \frac{1}{6}(W_{(k+1)\Delta} - W_{k\Delta})^3 + \frac{1}{2}(W_{(k+1)\Delta} - W_{k\Delta})\Delta \end{aligned}$$

We will need only the following representation of (4):

$$E[F|\mathscr{F}_{k\Delta}] = \sum_{j=0}^{\infty} \gamma_{k,j} E[D^j_{(k+1)\Delta}F|\mathscr{F}_{(k+1)\Delta}] \qquad (5)$$

where:

$$\gamma_{k,j} = \begin{cases} 1 & \text{if} \quad j=0 \\ (-1)^{j+1} g_{k,j} & \phantom{\text{if}} \quad j>0 \end{cases}$$

We rewrite (2) and (3) as:

$$\hat{C}_{m\Delta} \simeq E[\max(\hat{C}_{(m+1)\Delta}, h_{(m+1)\Delta})|\mathscr{F}^d_{m\Delta}] \qquad (6)$$

If we replace the maximum function by any smooth approximation, it is easy to see that the recursion (6) can be implemented by a recursive use of (5), where at each step m we calculate the conditional expectation, of $\max(\hat{C}_{(m+1)\Delta}, h_{(m+1)\Delta})$ instead of the conditional expectation of F. The remaining numerical problem is to find a suitable way to numerically compute the conditional expectation of the j-th Malliavin derivative of a functional F.

THE DNRT ALGORITHM

The DNRT algorithm consists in a particular choice of the computation of the conditional expectation of the j-th Malliavin derivative of a functional F, namely $E[D^j_{(m+1)\Delta}F|\mathscr{F}^d_{(m+1)\Delta}]$. Since the Malliavin derivative at time t commutes with the conditional expectation at time t, we have:

$$E[D^j_{(m+1)\Delta}F|\mathscr{F}^d_{(m+1)\Delta}] = D^j_{(m+1)\Delta}E[F|\mathscr{F}^d_{(m+1)\Delta}]$$

By definition, for any $\mathscr{F}^d_{M\Delta}$-measurable random variable X the Malliavin derivative is:

$$D^j_{(m+1)\Delta}X = \sum_{k>m} \frac{\partial X}{\partial W_{k\Delta}}$$

Since $E[F|\mathscr{F}^d_{(m+1)\Delta}]$ does not depend on $W_{k\Delta}$ for $k > m+1$, we have:

$$E[D^j_{(m+1)\Delta}F|\mathscr{F}^d_{(m+1)\Delta}] = \frac{\partial}{\partial W_{(m+1)\Delta}} E[F|\mathscr{F}^d_{(m+1)\Delta}]$$

Thus, we can rewrite (5) as:

$$E[F|\mathscr{F}_{m\Delta}] = \sum_{j=0}^{\infty} \gamma_{m,j} \frac{\partial^j}{\partial W^j_{(m+1)\Delta}} E[F|\mathscr{F}^d_{(m+1)\Delta}] \quad (7)$$

The American option recursion (6) becomes:

$$\hat{C}_{m\Delta} \simeq \sum_{j=0}^{J} \gamma_{m,j} \frac{\partial^j}{\partial W^j_{(m+1)\Delta}} \max(\hat{C}_{(m+1)\Delta}, h_{(m+1)\Delta}) \quad (8)$$

While the max function (8) is a problem if we take exact derivatives, our algorithm will consist in taking a numerical derivative of $\max(\hat{C}_{(m+1)\Delta}, h_{(m+1)\Delta})$. To that effect, we need some notation. For simplicity, we describe our algorithm only in the case $J = 2$, i.e., we take the first two derivatives of the conditional expectation.

Note first that recursion (8) is valid for any scenario ω. The value $\hat{C}(0,\omega)$ should thus be independent of the scenario ω. However, due to the truncation in the summation, $\hat{C}(0,\omega)$ will depend on the scenario. To that effect, for each scenario ω, we construct a tree of trinomial subscenarios. Each subscenario s over M timesteps is labeled by a sequence of $m = 1..M$ characters α_m where $\alpha_m \in \{D,C,U\}$. We set:

$$\left\{ \begin{array}{c} \alpha_m = U \\ \alpha_m = C \\ \alpha_m = D \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} W(m\Delta,\omega_s) = W((m-1)\Delta,\omega_s) + W(m,\omega) - W(m-1,\omega) + \varepsilon\Delta \\ W(m\Delta,\omega_s) = W((m-1)\Delta,\omega_s) + W(m,\omega) - W(m-1,\omega) \\ W(m\Delta,\omega_s) = W(i\Delta,\omega_s) + W(m,\omega) - W(m-1,\omega) - \varepsilon\Delta \end{array} \right\} \quad (9)$$

Thus:

$$s = \alpha_1 \alpha_2 .. \alpha_M$$

See figure 1 for an illustration.

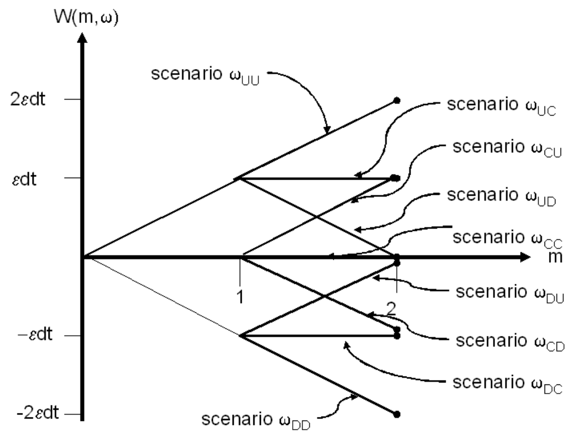

FIGURE 1. For the scenario ω consisting of $W(\Delta,\omega) = W(2\Delta,\omega) = 0$ we represent the 9 subscenarios in the differential non-recombing tree.

We call $\mathscr{S}_{\omega,m}$ the set of all scenarios $\{\omega_{\alpha_1..\alpha_M} | \alpha_1,..,\alpha_m \in \{D,C,U\}, \alpha_{m+1} = .. = \alpha_M = C\}$. Let us fix a particular subscenario, and call $\alpha_1..\alpha_M$ its list of characters. For that particular subscenario, we define the subscenario $\omega^D_{s,m}$ consisting of the characters $\alpha^D_1..\alpha^D_M$ with:

$$\left.\begin{array}{l} \alpha_i^D \\ \alpha_i^D \end{array}\right\} = \left\{ \begin{array}{l} \alpha_i \\ D \end{array} \right. \text{if} \begin{array}{l} i \neq m \\ i = m \end{array}$$

Likewise, we define the subscenario $\omega^U_{s,m}$ consisting of the characters $\alpha^U_1..\alpha^U_M$ with:

$$\left.\begin{array}{l} \alpha_i^U \\ \alpha_i^U \end{array}\right\} = \left\{ \begin{array}{l} \alpha_i \\ U \end{array} \right. \text{if} \begin{array}{l} i \neq m \\ i = m \end{array}$$

We can now approximate the Malliavin $D^j_{m\Delta} V(m\Delta,\omega_s)$ by numerical derivatives, which we denote by $\bar{D}_{m\Delta}(\omega_s)$.

Algorithm DNRT

(i) Simulate Ω independent paths $\{W(m\Delta,\omega)\}$ of Brownian motion, for $\omega = 1\ldots\Omega$, $m = 1\ldots M$.
(ii) For each scenario $\omega = 1..\Omega$ simulate subscenarios $\{W(m\Delta,\omega_s)\}$ for $\omega_s \in \mathscr{S}_{\omega,M}$ according to (9)
(iii) For all ω and all $\omega_s \in \mathscr{S}_{\omega,M}$ set:

$$V(M\Delta,\omega_s) = h(M\Delta,\omega_s)$$

(iv) Apply backward induction: for $m = M,..,1$ for all ω and all $\omega_s \in \mathscr{S}_{\omega,m}$ set

$$
\begin{aligned}
\bar{D}^0_{m\Delta}(\omega_s) &= V(m\Delta, \omega_s) \\
\bar{D}^1_{m\Delta}(\omega_s) &= \frac{V(m\Delta, \omega^U_{s,m}) - V(m\Delta, \omega^D_{s,m})}{2\varepsilon\Delta} \\
\bar{D}^2_{m\Delta}(\omega_s) &= \frac{V(m\Delta, \omega^U_{s,m}) - 2V(m\Delta, \omega_s) + V(m\Delta, \omega^D_{s,m})}{\varepsilon^2\Delta^2} \\
V((m-1)\Delta, \omega_s) &= \sum_{j=0}^{2} \gamma_{m-1,j}\bar{D}_{m\Delta}(\omega_s)
\end{aligned}
$$

(iv) Set

$$\hat{V}(0) = \frac{1}{\Omega}\sum_{\omega=1}^{\Omega} V(0, \omega)$$

RESULTS

We applied the DNRT algorithm to the pricing of an Asian option, namely an average price call. Exercise dates are at a multiple of $\Delta = 0.125$, so this makes this option in fact Bermudan. The expiration is $T = M\Delta = 1.125$ and the strike price is denoted K. We model the stock price S as a geometric Brownian motion. For simplicity, the interest rate $r = 0$, thus:

$$S(t) = S_0 \exp(-\frac{1}{2}\sigma^2 t + \sigma W(t))$$

We take $S_0 = 10$. The payoff at maturity T is thus:

$$h(T) = \max(\frac{1}{T}\sum S(t)\Delta - K, 0)$$

As explained earlier, we use a second order approximation, and use $\Omega = 10$ scenarios. We chose $\varepsilon = 5$. We vary the strike price K by increments of \$1 from 0 to \$10.

We calculate first the price of an European version of that option, and secondly of an American version of that option, and thus show $\hat{V}(0)$ as a function of the strike price in figure 2. For comparison purposes, we show the same results obtained by Monte Carlo simulation with 200 scenarios. The American option is valued by the Tsitsiklis-Van Roy algorithm, where we use as basis functions a constant plus the first four powers of Brownian motion at the time of exercise.

As expected the American option values are higher than the European option values, and the difference grows with the strike price but the difference is not very significant. This can be explained by two facts. First, the option is most of the time in-the-money. Secondly, we would have obtained better results for the Tsitsiklis-Van Roy algorithm had we employed basis function spanning the full path of Brownian motion before each exercise time. However, the complexity of such an algorithm was unmanageable, even with only 9 time steps.

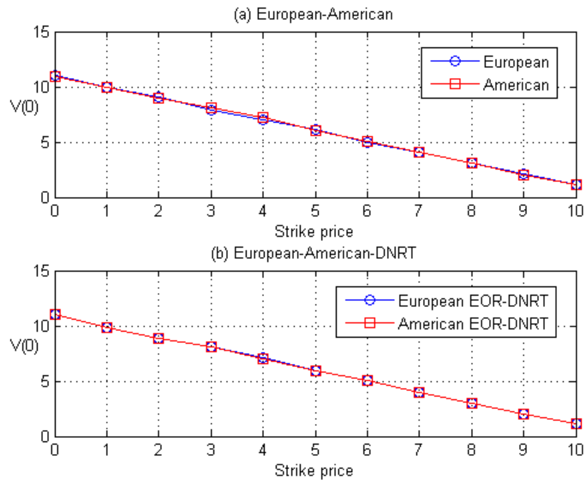

FIGURE 2. (a) The average prices for different European and American options.(b) The average prices by EOR-DNRT.

The difference between the average values of our DNRT estimator and the benchmarks grows with the strike price. Also, the coefficient of variation of our DNRT estimator grows with the strike price. This is easy to explain. The convexity of the option grows with the strike price, and a second order approximation becomes less and less accurate.

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