Exponential Formula For Poisson Process

Sixian.Jin, Henry Schellhorn

April 12, 2014

Proof. We consider the compensate Poisson process $M_t = N_t - \lambda t$ in following.
We follow the definition of Malliavin derivative as:

$$D_t I_n(f_n) = n I_{n-1}(f_{n-1}(\cdot, t))$$

with

$$I_n(f_n) = n! \int_0^T \int_0^{s_1} ... \int_0^{s_{n-1}} f_n(s_1, ..., s_n) dM_{s_1} ... dM_{s_n}.$$

It follows the property:

$$D(FG) = DF + DG + DFDG,$$

Following this definition the integration by part rule should be: if $F = F(M_T)$, then

$$\int_0^T F dM_s = F \int_0^T dM_s - \lambda \int_0^T D_s F ds - \int_0^T D_s F dM_s. \tag{1}$$

Suppose F is discrete like $F = F(M_T, M_t)$, following Clark-Ocone, we still have

$$E[F|\mathcal{F}_t] = \sum (-1)^n \int_t^T \int_{s_1}^T ... \int_{s_{n-1}}^T D_T F dM_{s_1} ... dM_{s_n}.$$

Firstly considering one integration and repeating (1), we get

$$\int_t^T F dM_{s_1} = \sum_{i=0}^{\infty} (-1)^i D_T^i F \int_t^T dM_s - \lambda \sum_{i=0}^{\infty} (-1)^i D_T^{i+1} F \int_t^T ds.$$

Then considering double integration and denote $G = \sum_{i=0}^{\infty} (-1)^i D_T^i F$,

$$\int_t^T \int_{s_1}^T F dM_{s_1} dM_{s_2} = \int_t^T G \left(\int_{s_1}^T dM_{s_2} \right) dM_{s_1} - \lambda \int_t^T D_T G \left(\int_{s_1}^T ds \right) dM_{s_1}$$

$$= \sum_{i=0}^{\infty} (-1)^i D_T^i G \int_t^T \int_{s_1}^T dM_{s_2} dM_{s_1} - \lambda \sum_{i=0}^{\infty} (-1)^i D_T^{i+1} G \int_t^T \int_{s_1}^T dM_{s_2} ds_1$$

$$-\lambda \sum_{i=0}^{\infty} (-1)^i D_T^i D_T G \int_t^T \left(\int_{s_1}^T ds_2 \right) dM_{s_1} + \lambda^2 \sum_{i=0}^{\infty} (-1)^i D_T^{i+1} D_T G \int_t^T \int_{s_1}^T ds_2 ds_1$$

$$= \sum_{i=0}^{\infty} (-1)^i D_T^i G \int_t^T \int_{s_1}^T dM_{s_2} dM_{s_1} - \lambda \sum_{i=0}^{\infty} (-1)^i D_T^{i+1} G \int_t^T ds_1 \int_t^T dM_{s_2}$$

$$+\lambda^2 \sum_{i=0}^{\infty} (-1)^i D_T^{i+2} G \int_t^T \int_{s_1}^T ds_2 ds_1$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}D_T^{i+j}F\int_t^T\int_{s_1}^T dM_{s_2}dM_{s_1} - \lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+1}D_T^{i+j+1}F\int_t^T ds_1\int_t^T dM_{s_2}$$

$$+\lambda^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+2}D_T^{i+j+2}F\int_t^T\int_{s_1}^T ds_2ds_1.$$

So by induction,

$$\int_t^T\int_{s_1}^T...\int_{s_{n-1}}^T FdM_{s_1}...dM_{s_n}$$

$$= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}...\sum_{i_n=0}^{\infty}(-1)^{i_1+..+i_n}D_T^{i_1+..+i_n}F\int_t^T\int_{s_1}^T...\int_{s_{n-1}}^T dM_{s_1}...dM_{s_n}$$

$$+(-1)\lambda\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}...\sum_{i_n=0}^{\infty}(-1)^{i_1+..+i_n+1}D_T^{i_1+..+i_n+1}F\int_t^T\int_{s_1}^T...\int_{s_{n-2}}^T dM_{s_1}...dM_{s_{n-1}}\int_t^T ds_1 + ...$$

$$+(-1)^n\lambda^n\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}...\sum_{i_n=0}^{\infty}(-1)^{i_1+..+i_n+n}D_T^{i_1+..+i_n+n}F\int_t^T\int_{s_1}^T...\int_{s_{n-1}}^T ds_1...ds_n$$

$$= \sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}...\sum_{i_n=0}^{\infty}\lambda^j(-1)^{i_1+..+i_n}D_T^{i_1+..+i_n+j}FJ_{n-j}(M_T-M_t)\frac{(T-t)^j}{j!}.$$

where I denote $J_{n-j}(M_T-M_t)=\int_t^T\int_{s_1}^T...\int_{s_{n-j-1}}^T dM_{s_1}...dM_{s_{n-j}}$ which is a Charlier polynomial .

So the backward Taylor expansion is

$$E[F|\mathcal{F}_t] = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}...\sum_{i_n=0}^{\infty}\lambda^j(-1)^{i_1+..+i_n+l}D_T^{i_1+..+i_n+j+l}FJ_{n-j}(M_T-M_t)\frac{(T-t)^j}{j!}.$$

■

Proof. Since we have proved

$$E[F|\mathcal{F}_t] = \sum_{n=0}^{\infty}\gamma(n)D_T^nF,$$

by applying Charlier polynomial with respect to M_T as test functions, we list first four terms,

$$C_0(M_t,t)=1, C_1(M_t,t)=M_t,$$

$$C_2(M_t,t)=M_t^2-2M_t-\lambda t,$$

$$C_3(M_t,t)=M_t^3-2M_t^2-3\lambda tM_t+2M_t+2\lambda t.$$

We get easily $\gamma(0)=1,\ \gamma(1)=M_t-M_T=C_1(M_t-M_T,t-T)$, then by $C_n(M_t,t)=I_n(1_{[0,t]^{\otimes n}})$,

$$E[C_2(M_T,T)|\mathcal{F}_t]=C_2(M_t,t)$$

$$=C_2(M_T,T)+2(M_t-M_T)C_1(M_T,T)+2\gamma(2)$$

so by directly computation we obtain,

$$\gamma(2)=\frac{1}{2}((M_t-M_T)^2-2(M_t-M_T)-2\lambda(t-T))=\frac{1}{2!}C_2(M_t-M_T,\lambda(t-T)).$$

Using directly computation to check one more term,

$$C_3(M_t,t) = C_3(M_T,T) + 3(M_t - M_T)C_2(M_T,T) + 3C_2(M_t - M_T, t-T)M_T + 3!\gamma(3)$$

replacing everything by known result and we obtain

$$\gamma(3) = \frac{1}{3!}((M_t - M_T)^3 - 3(M_t - M_T)^2 - 3\lambda(t-T)(M_t - M_T) + 2(M_t - M_T) + 2\lambda(t-T))$$

which is just $\frac{1}{3!}C_3(M_t - M_T, \lambda(t-T))$.

So by induction we obtain the conclusion

$$E[F|\mathcal{F}_t] = \sum_{n=0}^{\infty} \gamma(n) D_T^n F,$$

where

$$\gamma(n) = \frac{1}{n!}C_n(M_t - M_T, \lambda(t-T)).$$

■

Proof.
We know the set

$$\{e^{-\lambda \int_0^T u(s)ds} \prod_{s,\ N_s - N_{s-} \neq 0} (1 + u(s)),\ u \text{ is continuous}\}$$

is dense in $L^2(\Omega)$. I rewrite it by choose exponential and log for the product,

$$\varepsilon(u)_T = e^{-\lambda \int_0^T u(s)ds} e^{\int_0^T \log(1+u(s))dN(s)}.$$

So

$$E[\varepsilon(u)_T|\mathcal{F}_t] = \varepsilon(u)_t = e^{-\lambda \int_0^t u(s)ds} e^{\int_0^t \log(1+u(s))dN(s)}$$

while acting on "stop path" for Poisson process N_t, one obtain

$$\varepsilon(u)_T(\omega^t) = e^{-\lambda \int_0^T u(s)ds} e^{\int_0^t \log(1+u(s))dN(s)},$$

so we get

$$E[\varepsilon(u)_T|\mathcal{F}_t] = \varepsilon(u)_T(\omega^t) e^{\lambda \int_t^T u(s)ds}.$$

Then according to chain rule, i.e., $D_t\varepsilon(u)_T = u(t)\varepsilon(u)_T$, we obtain

$$E[\varepsilon(u)_T|\mathcal{F}_t] = \varepsilon(u)_T(\omega^t) + \lambda \int_t^T D_s\varepsilon(u)_T(\omega^t)ds + \lambda^2 \int_t^T \int_{s_1}^T D_{s_1}D_{s_2}\varepsilon(u)_T(\omega^t)ds_1 ds_2 + ...$$

It is easy to see λD_t is a linear operator so we obtain, for any $F \in L^2$,

$$E[F|\mathcal{F}_t] = F(\omega^t) + \lambda \int_t^T D_sF(\omega^t)ds + \lambda^2 \int_t^T \int_{s_1}^T D_{s_1}D_{s_2}F(\omega^t)ds_1 ds_2 + ...$$

■